Construction of time operators associated with Schrödinger operators

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December 1, 2016 PNU Math forum

Let $[\cdot,\cdot]$ be the commutator defined by

$$[A,B] = AB - BA$$

for linear operators A and B on a vector space D.

- **Example** 1: For d-dimensional matrices S and T,

$$[S,T] \neq O.$$

- **Example** 2: For differential operator ∂x and the multiplication operator x, since $\partial xxf = x\partial xf + f$, we have

$$[\partial x, x] = 1.$$

Let A and B be linear operators on a complex Hilbert space $\mathscr{H}$,

satisfying the canonical commutation relation (CCR)

$$[A,B] = -i1$$

on a dense subspace

$$\mathscr{D} \subset D(AB) \cap D(BA)$$

- We call $\mathscr{D}$ a CCR-domain for the pair (A,B).
- $(\mathscr{H}, \mathscr{D}, \{A,B\})$ is called a representation of CCR

Extension to d-degree system

Let A_j and B_j be linear operators on $\mathscr{H}$, and dense $\mathscr{D} \subset \mathscr{H}$ such that

$$\mathscr{D} \subset \cap_{j,k=1}^{d} [D(A_j B_k) \cap D(B_k A_j) \cap D(A_j A_k) \cap D(B_j B_k)].$$

Then

$$(\mathscr{H}, \mathscr{D}, \{A_j, B_j | j = 1, \ldots, d\})$$

is also called a representation of CCR if

$$[A_j, B_k] = -i\delta_{jk} 1, \quad [A_j, A_k] = 0 = [B_j, B_k]$$

hold on $\mathscr{D}$.

Example

$P_j = -i\partial x_j$, $Q_i = x_i\times$.
Example 1 $[P_j, Q_{j'}] = -i\delta_{jj'}1$, $1 \le j, j' \le d$
Example 2 $[-\frac{1}{2}\sum_j P_j^2, T_{AB}] = -i1$, where

$$T_{AB} = \frac{1}{2}\sum_j \left(\frac{1}{P_j}Q_j + Q_j\frac{1}{P_j}\right)$$

Example 3 $[f(P), T_f] = -i1$, where

$$T_f = \frac{1}{2}\sum_j \left(\frac{1}{\partial_j f(P)}Q_j + Q_j\frac{1}{\partial_j f(P)}\right)$$

Example 4 $[\sqrt{P^2+1}, T] = -i1$, $T = \cdots$.

One may infer that self-adjoint operator H may have a symmetric operator T corresponding to time, satisfying CCR

$$[H, T_H] = -i1.$$

Such an operator T_H is called a **time operator** of H.

Difficulty

Let $H\phi = E\phi$. Then $[H,T]\phi = (H-E)T\phi = -i\phi$ and

$$T\phi = -i(H-E)^{-1}\phi?$$

Thus $\phi \notin D(T)$ for any e.v. ϕ of H.

Hierarchy of time operators

[Weyl relation] A pair (A,B) consisting of self-adjoint operators A and B is called a weak Weyl representation if the **Weyl relations** holds:

$$e^{-itA}e^{-isB}\psi = e^{ist}e^{-isB}e^{-itA}\psi \quad t,s \in \mathbb{R}$$

[Weak Weyl relation] A pair (A,B) consisting of a self-adjoint operator A and symmetric operator B is called a **weak Weyl representation** if

$$e^{-itA}D(B) \subset D(B) \quad Be^{-itA}\psi = e^{-itA}(B+t)\psi$$

holds for all $\psi \in D(B)$ and all $t \in \mathbf{R}$.

Important

Weyl relation $\Longrightarrow$ weak Weyl relation $\Longrightarrow$CCR

Definition (Ultra-strong time operator)

A self-adjoint operator T is called a **ultra-strong time operator** of H if (H,T) is a Weyl representation.

Definition (Strong time operator)

A symmetric operator T is called a **strong time operator** of H on $\mathscr{H}$ if (H,T) is a weak Weyl representation.

[Remark1] Let $H > \infty$ and T be the strong time operator of H. Then T has **no self-adjoint extension**!
[Remark2] Let T be a strong time operator of H. Then $\sigma(H)$ is purely absolutely cont. In particular **if H has an eigenvalue, then H has no strong time operator.**

Definition (weak time operator)

A symmetric operator T is called a **weak time operator** of H if a dense subspace $\exists \mathscr{D}_{\rm w} \subset D(T) \cap D(H)$ s.t.

$$(H\phi, T\psi) - (T\phi, H\psi) = -i(\phi, \psi), \quad \phi, \psi \in \mathscr{D}_{\rm w}.$$

We call $\mathscr{D}_{\rm w}$ a weak-CCR domain for the pair (H, T).

Definition (Ultra-weak time operator)

Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be dense subspaces of $\mathscr{H}$. A sesquilinear form

$$\mathfrak{t} : \mathscr{D}_1 \times \mathscr{D}_2 \to \mathbf{C}$$

is called a **ultra-weak time operator** of H if dense subspaces $\exists\mathscr{D}, \exists\mathscr{E} \subset \mathscr{D}_1 \cap \mathscr{D}_2$ such that (i)–(iii) hold:

(i) $\mathscr{E} \subset D(H) \cap \mathscr{D}$,

(ii) $\mathfrak{t}[\phi,\psi]^* = \mathfrak{t}[\psi,\phi]$, $\phi,\psi \in \mathscr{D}$,

(iii) $H\mathscr{E} \subset \mathscr{D}_1$

$$\mathfrak{t}[H\phi,\psi] - \mathfrak{t}[H\psi,\phi]^* = -i(\phi,\psi) \quad \psi,\phi \in \mathscr{E}.$$

We call $\mathscr{E}$ an ultra-weak CCR-domain for $(H,\mathfrak{t})$ and $\mathscr{D}$ a symmetric domain of $\mathfrak{t}$.

Remark

Weak time operator $\subset$ ultra weak time operator

- Let T be a weak time operator of H.
- $\mathfrak{t}_T : \mathscr{H} \times D(T) \to \mathbf{C}$ defined by

$$\mathfrak{t}_T[\phi, \psi] = (\phi, T\psi), \quad \phi \in \mathscr{H}, \psi \in D(T).$$

- $\mathfrak{t}_T$ is an ultra-weak time operator of H.

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5 classes of time operators

ultra-strong time $\subset$ strong time $\subset$ time $\subset$ weak time $\subset$ **ultra-weak time**

Idea of constructing time operator of H

The purpose of my talk is to find ultra-weak time operator of

$$H = \frac{1}{2}\sum_{j=1}^{d} P_j^2 + V(Q) = H_{ac} \oplus H_{sc} \oplus H_p.$$

- T_{ac}: strong time operator of absolutely cont. part H_{ac} (scattering theory).
- $\mathfrak{t}_p$: ultra weak time operator of point spectrum part: H_p.
- The absence of singular cont. part: H_{sc}.

Then

$$T_{ac} \text{``}+\text{''} \mathfrak{t}_p$$

is an ultra weak time operator of H.

Absolutely continuous spectrum and strong time operators

Scattering theory

Let $H_0 = \frac{1}{2}\sum_j P_j^2$. Under some condition on V we have $H_0 = UH_{ac}U^{-1}$ with some unitary operator (wave operator) U.

$[H_0, T_{AB}] = -i1$

Theorem (Arai (06), Strong time operators)

$T'_{AB} = U^{-1}T_{AB}U$ *is a strong time operator of* H_{ac}*. I.e.,*

$$[H_{ac}, T'_{AB}] = -i1.$$

Discrete spectrum and ultra-weak time operators

How can we construct a time operator of H_p?

Galapon(02), Arai-Matsuzawa(08)

Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal bases. Suppose that $\sigma(H) = \{E_j\}_{j=1}^{\infty}$, every E_j is *simple*, and $\sum_{j=1}^{\infty} \frac{1}{E_j^2} < \infty$. Then

$$T\phi = i\sum_{n=1}^{\infty}\left(\sum_{m\neq n}\frac{(e_m,\phi)}{E_n - E_m}\right)e_n$$

is a time operator of H. I.e., $[H,T]\phi = -i\phi$.

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Ultra-weak time operator of the case $\sigma(H)=\sigma_{disc}(H)=\{E_n\}_{n=1}^\infty$

- **[$E_n\to\infty$]** Let $\lim_{n\to\infty}E_n=\infty$. Then **time operator** $\exists T$ of H.
- **[$E_n\to 0$]** Let $E_n<0$, $\lim_{n\to\infty}E_n=0$ and $0\notin\sigma_p(H)$. Then **ultra weak time operator** $\exists T$ of H.

Definition (class $S(\mathscr{H})$)

A self-adjoint operator H on $\mathscr{H}$ is said to be in the class $S(\mathscr{H})$ if it has the following properties (H.1)–(H.4):

(H.1) $\sigma_{\rm sc}(H) = \emptyset$.

(H.2) $\sigma_{ac}(H) = [0, \infty)$.

(H.3) $\sigma_{\rm disc}(H) = \sigma_p(H) = \{E_n\}_{n=1}^{\infty}$, $E_1 < E_2 < \cdots < 0$, $\lim_{n\to\infty} E_n = 0$ (hence $0 \notin \sigma_p(H)$).

(H.4) There exists a strong time operator T_{ac} of H_{ac} in $\mathscr{H}_{ac}(H)$.

Theorem (Arai-H.(16) Ultra-weak time operator of H)

Let $H \in S(\mathscr{H})$. Then $\exists \mathfrak{t}_H$ ultra-weak time operator

Proof: $H = H_p \oplus H_{ac}$.

- By (H.3), H_p has an **ultra-weak time op.** $\mathfrak{t}_p$ such that

$$\mathfrak{t}_p[H_p\phi, \psi] - \mathfrak{t}_p[\phi, H_p\psi] = -i(\phi, \psi), \quad \phi, \psi \in \exists \mathscr{E}_p.$$

- By (H.4), H_{ac} has a **strong time operator** T_{ac} such that

$$[H_{ac}, T_{ac}] = -i1$$

- $\mathfrak{t}_H : (\mathscr{H}_{ac}(H) \oplus \mathscr{D}_p) \times (D(T_{ac}) \oplus \mathscr{D}_p) \to \mathbf{C}$ by

$$\mathfrak{t}_H[\phi_1 \oplus \phi_2, \psi_1 \oplus \psi_2] = (\phi_1, T_{ac}\psi_1) + \mathfrak{t}_p[\phi_2, \psi_2].$$

Example (Agmon potential)

Let $d \geq 3$. Suppose that $U \in L^\infty(\mathbf{R}^3)$. Then

$$V(x) = \frac{U(x)}{(1+|x|^2)^{1/2+\varepsilon}}$$

is an Agmon potential for all $\varepsilon > 0$. Suppose that U is negative, continuous, spherically symmetric and satisfies that $U(x) = -1/|x|^\alpha$ for $|x| > R$ with $0 < \alpha < 1$ and $R > 0$. Let $2\varepsilon + \alpha < 2$. Then H has an ultra-weak time operator.

Example (hydrogen atom)

The hydrogen Schrödinger operator $H_{\rm hyd} = -\Delta - \gamma/|x|$ is self-adjoint with $D(H_{\rm hyd}) = D(H_0)$. The Coulomb potential $-\gamma/|x|$ with $d = 3$ is not an Agmon potential. But we can show that $H_{\rm hyd}$ has an ultra-weak time operator.

Summary

(1) We construct an ultra-weak time operator $\mathfrak{t}$ of $H = \frac{1}{2}P^2 + V(Q)$:

$$\mathfrak{t}(H\phi, \psi) - \mathfrak{t}(H\psi, \phi)^* = -i(\phi, \psi)$$

(2) $\mathfrak{t}$ is **densely** defined.
(3-1) We assume $\sigma_{sc}(H) = \emptyset$ and $0 \notin \sigma_p(H)$.
(3-2) We assume $\#\sigma_{disc}(H) = \infty$ or $\#\sigma_{disc}(H) = 0$.
(4) H_{hyd} is included in our results.

Reference: A. Arai and F. Hiroshima, Ultra-Weak Time Operators of Schrödinger Operators, arXiv:1607.04702, 2016